INITIAL VALUE PROBLEMS FOR SYSTEMS OF ORDINARY FIRST AND SECOND ORDER DIFFERENTIAL EQUATIONS WITH A SINGULARITY OF THE FIRST KIND*

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Abstract. Analytical properties like existence, uniqueness and smoothness of bounded solutions of nonlinear singular initial value problems for ordinary differential equations of first and second order are considered. Particular attention is paid to the structure of initial conditions which are necessary and sufficient for the solution to be continuous.

Key words. Ordinary differential equations, initial value problems, singularity of the first kind, existence and uniqueness theory.

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1. Introduction. We investigate two classes of nonlinear initial value problems, first order systems of the form

$$y'(t) = \frac{M(t)}{t}y(t) + f(t, y(t)), \quad t \in (0, 1], \tag{1.1a}$$

$$B_0 y(0) = \beta, \tag{1.1b}$$

$$y \in C[0, 1], \tag{1.1c}$$

where y, f are vector-valued functions of dimension n, M is an $n \times n$ matrix, B_0 is an $m \times n$ matrix and β is a vector of dimension $m \le n$, and second order systems of the form

$$y''(t) = \frac{A_1(t)}{t}y'(t) + \frac{A_0(t)}{t^2}y(t) + f(t, y(t)), \quad t \in (0, 1], \tag{1.2a}$$

$$B_0 y(0) = \beta, \tag{1.2b}$$

$$y \in C[0, 1], \tag{1.2c}$$

where A_0, A_1 are $n \times n$ matrices, B_0 is an $m \times n$ matrix and β is a vector of dimension $m \le n$. In both cases we also will consider "terminal value problems" where the boundary conditions are posed at $t = 1$ instead of $t = 0$.

Research activities in the field of singular problems as well as in related areas (see enclosed list of publications) are a strong motivation for the search for a method to be used as a basis for a reliable *standard code* designed especially for solving *singular boundary value problems*, and taking into account the specific difficulties caused by singularities. Our aim is to develop a theoretical background for a shooting procedure based on the numerical solution of *singular initial value problems* and this paper provides the study of the analytical properties of (1.1) and (1.2) which have to be

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examined before beginning the convergence analysis of underlying one-step or multistep method. Here, existence and uniqueness of bounded solutions, as well as their smoothness, will be discussed.

Because of numerous applications occurring in the form of singular boundary value problems[1] this class of singular systems has been extensively studied, while there are only very few papers dealing with the approximation of initial value problems (see [10], [12] and [15]). Moreover, as far as we are aware, there is no framework discussing the solvability of (1.1) and (1.2) although this theory is quite important in the context mentioned above.

Various mathematical models of applications from physics and chemistry take the form of systems of time-dependent partial differential equations subject to initial/boundary conditions. For the investigation of stationary solutions many of these models can be reduced to singular systems of ODEs of second order, especially when - due to symmetries in the geometry of the problem data - polar, cylindrical or spherical coordinates can be used (see the Thomas-Fermi equation, [4], [8], [17]) occurring in problems from quantum mechanics and astrophysics and the Ginzburg-Landau equation ([21], [29]), describing ferromagnetic systems and arising in superconductivity models. Further examples are singular Sturm-Liouville eigenvalue problems ([1], [26]), problems from chemical reactor theory ([24], [28]) and applications from mechanics, especially from buckling of spherical shells (see [6], [7], [14], [28]).
Such problems have been solved numerically using standard methods in [13] (finite difference schemes), [14] (Runge-Kutta and Adams predictor-corrector methods), [22] (generalized spline methods), and [28] (multiple shooting and Taylor series) and their successful application in numerical simulation was followed (for scalar problems) by investigations of stability and convergence properties in [13], [23] (finite difference schemes) and [25], [27], [31] (collocation and finite differences). A first contribution to the analysis of singular first order systems has been made by Brabston and Keller in [2] and [3] and Natterer in [22]. In [9] and [11] the existence of bounded solutions, finite difference methods and collocation methods have been investigated by de Hoog and Weiss. The techniques developed in these papers have been modified to handle singular systems of second order, both linear and nonlinear (see [32]–[35]).

An outline of the paper is: §3 deals with analytical questions for first order systems and is subdivided into three parts for linear problems with constant coefficient matrix M, linear problems with variable coefficient matrix $M(t)$, and the nonlinear case. In §4 the analogous results for the second order problems are formulated.
Throughout the paper we heavily rely on techniques developed in [9] and [32], and refer for details to these papers. Particular attention is paid to the structure of the initial conditions necessary to satisfy the smoothness requirements (1.1c) and to the discussion of resulting consequences in the structure of (1.1b).

2. Preliminaries. The following notation will be used. We denote by $\mathbb{C}^n$ the space of complex-valued vectors of dimension n and use $|\cdot|$ to denote the maximum norm in $\mathbb{C}^n$,

$$|x| = |(x_1, x_2, \cdots, x_n)^T| = \max_{1\le i\le n} |x_i|.$$

[1]See (1.1) and (1.2), where (1.1b) and (1.2b) are replaced by $b(y(0), y(1)) = 0$ and $b(y(0), y'(0), y(1), y'(1)) = 0$, respectively, b being an appropriately defined nonlinear vector-valued function.

$C^p_n[0,1]$ is the space of complex vector-valued functions which are p times continuously differentiable on $[0,1]$, and $C^p_n(0,1]$ is defined analogously. For a function $y \in C^0_n[0,1]$ we define the maximum norm

$$\|y\| = \max_{0\le t\le 1} |y(t)|.$$

We will also use the maximum norm restricted to the interval $[0,\delta]$, $\delta > 0$,

$$\|y\|_\delta = \max_{0\le t\le \delta} |y(t)|.$$

$C^p_{n\times n}[0,1]$ is the space of complex-valued $n\times n$ matrices with columns from $C^p_n[0,1]$. For a matrix $A = (a_{ij})^n_{i,j=1}$, $A \in C^0_{n\times n}[0,1]$, $\|A\|$ is the induced norm,

$$\|A\| = \max_{0\le t\le 1} |A(t)| = \max_{0\le t\le 1}\left(\max_{1\le i\le n}\sum_{j=1}^n |a_{ij}(t)|\right).$$

Also for matrices, we occasionally will use the maximum norm $\|A\|_\delta$ on $[0,\delta]$, $\delta > 0$. Where there is no confusion we will delete the subscripts n and $n\times n$ and call $C = C[0,1] = C^0[0,1]$, $C(0,1] = C^0(0,1]$. For a $2n\times 2n$ identity matrix I we introduce the notation I_1 for the $n\times 2n$ matrix consisting of the n upper rows of I and I_2 for the $n\times 2n$ matrix consisting of the n lower rows of I.

3. Analytic results for systems of first order. We first study initial value problems of the form

(3.1a) $$y'(t) = \frac{M(t)}{t}y(t) + f(t,y(t)), \quad t\in(0,1],$$

(3.1b) $$B_0 y(0) = \beta,$$

(3.1c) $$y \in C[0,1],$$

where B_0 is an $m\times n$ matrix and β is a vector of dimension $m \le n$, and

(3.2a) $$y'(t) = \frac{M(t)}{t}y(t) + f(t,y(t)), \quad t\in(0,1],$$

(3.2b) $$B_1 y(1) = \beta,$$

(3.2c) $$y \in C[0,1],$$

where B_1 is an $n\times n$ matrix and β is a vector of dimension n.

3.1. Linear problems with constant coefficient matrix M. Consider the problems

(3.3a) $$y'(t) = \frac{M}{t}y(t) + f(t), \quad t\in(0,1],$$

(3.3b) $$B_0 y(0) = \beta,$$

(3.3c) $$y \in C[0,1],$$

and

(3.4a) $$y'(t) = \frac{M}{t}y(t) + f(t), \quad t\in(0,1],$$

(3.4b) $$B_1 y(1) = \beta,$$

(3.4c) $$y \in C[0,1].$$

The aim of this section is to formulate most general initial conditions which are necessary and sufficient for the solution y to be continuous on $[0,1]$.
We first construct the general solution of

$$y'(t) = \frac{M}{t} y(t) + f(t). \tag{3.5}$$

Let us denote by J the Jordan canonical form of M and by E the associated matrix of generalized eigenvectors of M. Moreover, let

$$v(t) := E^{-1} y(t),$$
$$g(t) := E^{-1} f(t).$$

Then we can rewrite (3.5) and obtain

$$v'(t) = \frac{J}{t} v(t) + g(t). \tag{3.6}$$

To simplify matters, we assume $J \in \mathbb{C}^{n \times n}$ to consist of only one box,

$$J = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \lambda & 1 \\ 0 & & & \lambda \end{pmatrix}, \quad \lambda = \sigma + \mathrm{i}\rho \in \mathbb{C}. \tag{3.7}$$

LEMMA 3.1. *Every solution of* (3.6) *has the form*

$$v(t) = \Phi(t) c + \Phi(t) \int_1^t \Phi^{-1}(\tau) g(\tau) \, d\tau, \tag{3.8}$$

where $c \in \mathbb{C}^n$ *is an arbitrary vector and*

$$\Phi(t) = t^J := \exp(J \ln(t))$$

is the fundamental solution matrix which satisfies

$$\Phi'(t) = \frac{J}{t} \Phi(t), \quad \Phi(1) = I, \quad t \in (0,1]. \tag{3.9}$$

Proof. See [5]. □

We note that the fundamental solution matrix has the form

$$t^J = t^\lambda \begin{pmatrix} 1 & \ln(t) & \frac{\ln(t)^2}{2} & \cdots & \frac{\ln(t)^{n-1}}{(n-1)!} \\ 0 & 1 & \ln(t) & \cdots & \frac{\ln(t)^{n-2}}{(n-2)!} \\ 0 & \ddots & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ln(t) \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}. \tag{3.10}$$

From the structure of this matrix and from Lemma 3.1 it is clear that the solution $v(t)$ given by (3.8) is not continuous on $[0,1]$ in general, and its smoothness depends on the eigenvalues of M. Consequently, we treat separately the cases[2] $\sigma < 0$, $\lambda = 0$, $\sigma > 0$.

Eigenvalues with negative real parts. Before formulating the main result of this section, we state the following lemma.

LEMMA 3.2. *Let $\gamma > 0$ and in J from (3.7) assume either $\sigma < 0$ or $\lambda = 0$. Then for*

$$u(t) := t^\gamma \int_0^1 s^{-J} s^{\gamma-1} f(st)\, ds$$

the following estimate holds:

$$|u(t)| \le \text{const.}\, t^\gamma \|f\|_\delta, \quad t \in [0,\delta].$$

Proof. Clearly,

$$|s^{-J}| \le s^{-\sigma} \sum_{i=0}^{n-1} \frac{|\ln(s)|^i}{i!}$$

and hence

$$\int_0^1 s^{\gamma-1} |s^{-J}|\, ds \le \int_0^1 \sum_{i=0}^{n-1} s^{\gamma-\sigma-1} \frac{(-\ln(s))^i}{i!}\, ds = \sum_{i=0}^{n-1} \sum_{k=0}^{i} \left. \frac{s^{\gamma-\sigma}(-\ln(s))^k}{k!(\gamma-\sigma)^{i+1-k}} \right|_0^1$$
$$= \sum_{i=0}^{n-1} \frac{1}{(\gamma-\sigma)^{i+1}} < \infty. \qquad \square$$

LEMMA 3.3. *Let all eigenvalues of M have negative real parts. Then for every $f \in C^p[0,1]$, $p \ge 0$, there exists a unique solution $y \in C$ of* (3.3a). *This solution has the form*

$$y(t) = t \int_0^1 s^{-M} f(st)\, ds, \tag{3.11}$$

and satisfies[3] $y(0) = 0$. *Moreover, $y \in C^{p+1}[0,1]$ and the following estimates hold:*

$$|y(t)| \le \text{const.}\, t\|f\|, \tag{3.12}$$

$$|y'(t)| \le \text{const.}\, \|f\|. \tag{3.13}$$

Proof. We first rewrite (3.8) and obtain[4]

$$v(t) = t^J \left(c - \int_0^1 \tau^{-J} g(\tau)\, d\tau \right) + t^J \int_0^t \tau^{-J} g(\tau)\, d\tau =: v_h(t) + v_p(t). \tag{3.14}$$

[2] We exclude the case of purely imaginary eigenvalues of M, $\lambda = \mathrm{i}\sigma$, leading to solutions of the form $t^{\mathrm{i}\sigma} = \cos(\sigma \ln t) + \mathrm{i} \sin(\sigma \ln t)$.

[3] This condition is necessary and sufficient for y to be continuous or equivalently, $y \in C[0,1]$ iff $y(0) = 0$.

[4] Here, J may consist of more than one Jordan box.

Change of variable in the second integral,

$$\tau \mapsto s = \frac{\tau}{t}, \tag{3.15}$$

yields

$$v_p(t) = t^J \int_0^t \tau^{-J} g(\tau)\, d\tau = t \int_0^1 s^{-J} g(st)\, ds. \tag{3.16}$$

Since the function $G(t,s) := s^{-J} g(st)$ is continuous on $[0,1] \times [0,1]$, $v_p \in C$ follows, and it is clear from (3.10) that $v \in C$ iff

$$c - \int_0^1 \tau^{-J} g(\tau)\, d\tau = 0.$$

Thus the unique solution $y \in C$ satisfying (3.3a) is

$$y(t) = t \int_0^1 s^{-M} f(st)\, ds =: t\eta(t). \tag{3.17}$$

This solution satisfies $y(0) = 0$ and there are no additional conditions of the form (3.3b) to be imposed.
We now examine the smoothness of y. We substitute (3.17) into (3.3a) and obtain

$$y'(t) = M\eta(t) + f(t).$$

The smoothness result follows on noting that $\eta \in C^p$ if $f \in C^p$, see (3.17). The estimate for $y(t)$ can be derived from Lemma 3.2 for $\gamma = \delta = 1$, and the estimate for $y'(t)$ follows by substituting (3.11) into (3.3a). □

We recapitulate: The results of Lemma 3.3 imply that in the case where all eigenvalues of M have negative real parts, the initial value problem (3.3) reduces to

$$y'(t) = \frac{M}{t} y(t) + f(t), \quad t \in (0,1],$$
$$y(0) = 0,$$

and $y \in C^{p+1}$ for any $f \in C^p$.

Eigenvalue $\lambda = 0$. Let X_0 be the eigenspace of M associated with the eigenvalue $\lambda = 0$, and let us denote by R the orthogonal projection onto X_0. Also, define

$$H := I - R.$$

Let $\tilde{R} \in \mathbb{C}^{n \times r}$ be the matrix consisting of the linearly independent columns of R, and for any mapping P denote by $\mathcal{R}(P)$ the range of P. In order to simplify the subsequent analysis we select a basis in which M is reduced to Jordan form and use this basis to construct the projections.

LEMMA 3.4. *Let all eigenvalues of M be zero. Then for every $f \in C^p[0,1]$, $p \geq 0$, and every $\gamma \in \mathcal{R}(R)$, there exists a solution $y \in C$ of* (3.3a)*. This solution has the form*

$$y(t) = \gamma + t \int_0^1 s^{-M} f(st)\, ds \tag{3.18}$$

and satisfies $My(0) = 0$. *Let* $m = r$ *and assume that the* $r \times r$ *matrix* $B_0\tilde{R}$ *is nonsingular. Then there exists a unique solution* $y \in C$ *satisfying* (3.3). *This solution is given by* (3.18) *with* $\gamma = \tilde{R}(B_0\tilde{R})^{-1}\beta$. *Moreover,* $y \in C^{p+1}[0,1]$ *and the following estimates hold:*

$$|y(t)| \le \text{const.}\, t\|f\| + |\tilde{R}(B_0\tilde{R})^{-1}\beta|, \tag{3.19}$$

$$|y'(t)| \le \text{const.}\, \|f\|. \tag{3.20}$$

Proof. We first consider (3.14),

$$v(t) = t^J\left(c - \int_0^1 \tau^{-J} g(\tau)\, d\tau\right) + t\int_0^1 s^{-J} g(st)\, ds =: v_h(t) + v_p(t)$$

and show that $v_p(t)$ is continuous. Define functions

$$h_m(t) := \int_{\frac{1}{m}}^1 s^{-J} g(st)\, ds, \quad m \in \mathbb{N},$$

and

$$h_\infty(t) := \int_0^1 s^{-J} g(st)\, ds.$$

Then, see the proof of Lemma 3.2,

$$\begin{aligned}\lim_{m\to\infty} |h_\infty(t) - h_m(t)| &= \lim_{m\to\infty}\left|\int_0^{\frac{1}{m}} s^{-J} g(st)\, ds\right| \\ &\le \lim_{m\to\infty} \text{const.} \sum_{i=0}^{n-1}\sum_{k=0}^{i} \frac{\frac{1}{m}|\ln(m)|^k}{k!} = 0\end{aligned}$$

and hence h_∞ is continuous as the uniform limit of continuous functions. Consequently, $v_p(t) = t h_\infty(t) \in C[0,1]$.

Let J consist of one Jordan box. Then it is obvious from (3.10) that $v \in C$ iff

$$J\left(c - \int_0^1 \tau^{-J} g(\tau)\, d\tau\right) = 0$$

and therefore v has the form

$$v(t) = c_1 u_1 + t\int_0^1 s^{-J} g(st)\, ds,$$

where $c_1 \in \mathbb{C}$ is an arbitrary constant and $u_1 \in \mathbb{R}^n$ is the first column of the identity matrix. This implies $My(0) = 0$ and

$$y(t) = c_1 e_1 + t\int_0^1 s^{-M} f(st)\, ds,$$

where e_1 is the first column of E and the only eigenvector of M. The generalization for an arbitrary J is clear and yields (3.18) with $y(0) = \gamma \in \ker M = \mathcal{R}(R)$. Consequently,

$My(0) = 0$ is a necessary and sufficient condition for the solution of (3.3a) to be continuous. The smoothness results for the solution follow immediately from the smoothness of f and (3.18).

Finally, we derive a condition which is sufficient for y from (3.18) to satisfy (3.3), or more precisely (3.3b). We substitute y into (3.3b) and obtain $B_0\gamma = \beta$. This yields

$$B_0\tilde{R}\alpha = \beta$$

on noting that for every $\gamma \in \mathcal{R}(R)$ there exists a unique $\alpha \in \mathbb{C}^r$ such that $\gamma = \tilde{R}\alpha$. The above system of m linear equations for α is uniquely solvable iff $m = r$ and $B_0\tilde{R}$ is nonsingular. In that case,

$$\gamma = \tilde{R}(B_0\tilde{R})^{-1}\beta.$$

The estimates (3.19) and (3.20) can be shownusing the results of Lemma 3.2 for $\lambda = 0$.
□

We now summarize the last result: If all eigenvalues of M are zero then (3.3) reduces to

$$\begin{aligned}
&y'(t) = \frac{M}{t}y(t) + f(t), \quad t \in (0,1],\\
&B_0y(0) = \beta,\\
&My(0) = 0,
\end{aligned}$$

and $y \in C^{p+1}$ for any $f \in C^p$. The initial conditions $My(0) = 0$ are necessary for $y \in C$ and can be equivalently expressed by $Hy(0) = 0$, $y(0) \in \ker M$ or $y(0) = Ry(0)$. They contain $n-r$ linearly independent equations. The remaining r equations necessary for the uniqueness of y are given by $B_0y(0) = \beta$.

We now consider (3.4). Since in this case all initial conditions need to be posed at $t = 1$ the only possible form of M is limited[5] to $M = 0$. This case is not interesting in its own right, but in the case of a matrix with more complex spectral properties (see Assumption A.3.2.) the next (fully obvious) lemma describes for the linear, decoupled problem those components of the solution which correspond to the eigenvalue $\lambda = 0$.

LEMMA 3.5. *Let $M = 0$ and the matrix B_1 be nonsingular. Then for every $f \in C^p[0,1]$, $p \geq 0$, and every $\beta \in \mathbb{C}^n$, there exists a unique solution $y \in C^{p+1}$ of* (3.4). *This solution has the form*

$$y(t) = B_1^{-1}\beta + \int_1^t f(\tau)\,d\tau,$$

and the following estimates hold:

$$|y(t)| \leq (1-t)\|f\| + |B_1^{-1}\beta|, \tag{3.21}$$

$$|y'(t)| \leq \|f\|. \tag{3.22}$$

[5]To see this consider the structure of the solution v from (3.14) for $n = 2$: $v_1(t) = c_1 + c_2\ln(t) + v_{p,1}(t)$, $v_2(t) = c_2 + v_{p,2}(t)$ with $v_1(1) = c_1 + v_{p,1}(1)$, $v_2(1) = c_2 + v_{p,2}(1)$. Clearly, $c_2 = 0$ is required for $v \in C$ which is equivalent to the initial condition $v_2(1) = v_{p,2}(1)$, but the value of the particular solution $v_{p,2}(1)$ is unknown, in general.

Eigenvalues with positive real parts.

LEMMA 3.6. *Let $\gamma \geq 0$ and in J from (3.7) assume $\sigma > 0$. Then for*

$$u(t) := \int_t^\delta \left(\frac{t}{\tau}\right)^J \tau^{\gamma-1}\, d\tau, \quad t \in [0,\delta]$$

the following estimates hold:

$$|u(t)| \leq \begin{cases} \text{const.} \left(\frac{t}{\delta}\right)^\sigma \left(1 + \left|\ln\left(\frac{t}{\delta}\right)\right|^{n-1}\right) \delta^\gamma, & \sigma < \gamma, \\ \text{const.}\, t^\sigma \left(1 + \left|\ln\left(\frac{t}{\delta}\right)\right|^{n}\right), & \sigma = \gamma, \\ \text{const.}\, t^\gamma, & \sigma > \gamma. \end{cases}$$

Proof. We treat separately the cases[6] $\sigma < \gamma$, $\sigma = \gamma$ and $\sigma > \gamma$.

(i) $\sigma < \gamma$:

$$\begin{aligned} |u(t)| \leq \int_t^\delta \left|\left(\frac{t}{\tau}\right)^J\right| \tau^{\gamma-1}\, d\tau &\leq \text{const.} \int_t^\delta \left(\frac{t}{\tau}\right)^\sigma \left(1 + \left|\ln\left(\frac{t}{\tau}\right)\right|^{n-1}\right) \tau^{\gamma-1}\, d\tau \\ &\leq \text{const.}\, t^\sigma \left(1 + \left|\ln\left(\frac{t}{\delta}\right)\right|^{n-1}\right) \int_t^\delta \tau^{\gamma-\sigma-1}\, d\tau \\ &= \text{const.}\, t^\sigma \left(1 + \left|\ln\left(\frac{t}{\delta}\right)\right|^{n-1}\right) (\delta^{\gamma-\sigma} - t^{\gamma-\sigma}). \end{aligned}$$

(ii) $\sigma = \gamma$:

$$\begin{aligned} |u(t)| \leq \int_t^\delta \left|\left(\frac{t}{\tau}\right)^J\right| \tau^{\gamma-1}\, d\tau &\leq \text{const.}\, t^\sigma \left(1 + \left|\ln\left(\frac{t}{\delta}\right)\right|^{n-1}\right) \int_t^\delta \tau^{-1}\, d\tau \\ &\leq \text{const.}\, t^\sigma \left(1 + \left|\ln\left(\frac{t}{\delta}\right)\right|^{n-1}\right) \left|\ln\left(\frac{t}{\delta}\right)\right|. \end{aligned}$$

(iii) $\sigma > \gamma$: In this case there exists an $\varepsilon > 0$ such that $\sigma = \gamma + 2\varepsilon$ and hence the result follows from

$$\begin{aligned} |u(t)| &\leq \int_t^\delta \left|\left(\frac{t}{\tau}\right)^J\right| \tau^{\gamma-1}\, d\tau \\ &\leq \text{const.} \int_t^\delta \left(\frac{t}{\tau}\right)^{\gamma+\varepsilon} \left[\left(\frac{t}{\tau}\right)^\varepsilon \left(1 + \left|\ln\left(\frac{t}{\tau}\right)\right|^{n-1}\right)\right] \tau^{\gamma-1}\, d\tau \\ &\leq \text{const.} \int_t^\delta \left(\frac{t}{\tau}\right)^{\gamma+\varepsilon} \tau^{\gamma-1}\, d\tau, \end{aligned}$$

on noting that the function $s^\varepsilon \left(1 + |\ln s|^{n-1}\right)$ is continuous on $[0,1]$. □

[6] For $\gamma = 0$ only the third case is relevant.

We now use the above result to prove the following lemma.

LEMMA 3.7. *Let all eigenvalues of M have positive real parts. Then for every $f \in C^p[0,1]$, $p \geq 0$, and every $c \in \mathbb{C}^n$, there exists a solution $y \in C$ of* (3.4a). *This solution has the form*

$$y(t) = t^M c + t^M \int_1^t \tau^{-M} f(\tau)\, d\tau =: y_h(t) + y_p(t). \tag{3.23}$$

If the matrix B_1 is nonsingular, then there exists a unique solution of (3.4). *This solution is given by* (3.23) *with $c = B_1^{-1}\beta$, and the following estimates hold:*

$$|y(t)| \leq \begin{cases} \text{const.}\, t^{\sigma_+}(1 + |\ln(t)|^{n_{\max}-1})(|B_1^{-1}\beta| + \|f\|), & \sigma_+ < 1, \\ \text{const.}\, t(1 + |\ln(t)|^{n_{\max}})(|B_1^{-1}\beta| + \|f\|), & \sigma_+ = 1, \\ \text{const.}\, t(|B_1^{-1}\beta| + \|f\|), & \sigma_+ > 1, \end{cases} \tag{3.24}$$

$$|y'(t)| \leq \begin{cases} \text{const.}\, t^{\sigma_+-1}(1 + |\ln(t)|^{n_{\max}-1})(|B_1^{-1}\beta| + \|f\|), & \sigma_+ < 1, \\ \text{const.}\, (1 + |\ln(t)|^{n_{\max}})(|B_1^{-1}\beta| + \|f\|), & \sigma_+ = 1, \\ \text{const.}\, (|B_1^{-1}\beta| + \|f\|), & \sigma_+ > 1, \end{cases} \tag{3.25}$$

where σ_+ is the smallest of the positive real parts of the eigenvalues of M and $n_{\max}$ is the dimension of the largest Jordan box in the Jordan canonical form J associated with M. This solution satisfies $y \in C[0,1] \cap C^{p+1}(0,1]$. Moreover, if $p < \sigma_+ \leq p+1$, then $y \in C^p[0,1] \cap C^{p+1}(0,1]$ and if $\sigma_+ > p+1$, then $y \in C^{p+1}[0,1]$.

Proof. Clearly, $y_h(t) \in C[0,1]$, see (3.10). Also, $y_p(t) \in C[\varepsilon,1]$, for any $\varepsilon > 0$, and the limit $\lim_{\varepsilon\to 0} y_p(\varepsilon)$ exists according to Lemma 3.6, where $\delta = \gamma = 1$. Therefore, $y \in C$. The estimate (3.24) follows immediately from

$$|t^M| \leq \text{const.}\, t^{\sigma_+}(1 + |\ln(t)|^{n_{\max}-1})$$

and Lemma 3.6, $\gamma = \delta = 1$. The estimate (3.25) follows by substituting $y(t)$ into (3.4a) and using (3.24).

We now discuss the smoothness properties of y. It is clear from

$$y_h^{(r)}(t) = \left(\frac{d^r}{dt^r} t^M\right) c = M^r t^{M-rI} c, \quad r = 1, \ldots, p+1,$$

that $p < \sigma_+ \leq p+1$ is sufficient for $y_h \in C^p[0,1] \cap C^{p+1}(0,1]$ and $\sigma_+ > p+1$ for $y_h \in C^{p+1}[0,1]$.

We now consider $y_p(t)$. The substitution of $y_p(t)$ into (3.4a) yields

$$y_p'(t) = M t^{M-I} \int_1^t \tau^{-M} f(\tau)\, d\tau + f(t) = M \int_1^t \left(\frac{t}{\tau}\right)^{M-I} \tau^{-1} f(\tau)\, d\tau + f(t)$$

and it follows immediately from Lemma 3.6 that $y_p \in C^1$ if $f \in C$ and $\sigma_+ > 1$. We use integration by parts to rewrite the above representation for y_p',

$$\begin{aligned} y_p'(t) &= M(I-M)^{-1} f(t) - M t^{M-I} (I-M)^{-1} f(1) \\ &\quad - M t^{M-I} \int_1^t \tau^{I-M} (I-M)^{-1} f'(\tau)\, d\tau + f(t), \end{aligned}$$

and from the differentiation thereof we have[7]

$$y_p''(t) = Mt^{M-2I}\left(f(1) + \int_1^t \tau^{I-M} f'(\tau)\,d\tau\right) + f'(t).$$

Consequently, if $\sigma_+ > 2$ and $f \in C^1[0,1]$, then $y_p \in C^2[0,1]$. Continuing this process we obtain analogous expressions for higher derivatives of y,

$$|y_p^{(p+1)}(t)| = Mt^{M-(p+1)I}\left(\int_1^t \tau^{pI-M} f^{(p)}(\tau)\,d\tau + \sum_{k=0}^{p-1}\prod_{l=p}^{k+2}(M-lI)f^{(k)}(1)\right) + f^{(p)}(t).$$

For $\sigma_+ > p+1$ and $f \in C^p[0,1]$, $y \in C^{p+1}[0,1]$ and the result follows. □

General systems. Before discussing the general case we introduce the following notation. Let X_+ be the invariant subspace associated with the eigenvalues with positive real parts, and let S be a projection onto X_+. Let P be a projection onto $X_0 \oplus X_+$. Define

$$P := R + S$$

and

$$Q := I - P.$$

The above discussion of the structure of smooth solutions suggests to associate different spectral properties of M with (3.3) and (3.4), respectively. For the subsequent investigations we therefore make the following assumptions.

- A.3.1 In (3.3) we assume all real parts of the eigenvalues of M to be nonpositive[8].
- A.3.2 In (3.4) we assume all real parts of the eigenvalues of M to be nonnegative. If zero is an eigenvalue of M, then the associated invariant subspace is assumed to be the eigenspace of M.

The following results stated without proofs are simple consequences of Lemmas 3.3, 3.4, 3.5 and 3.7.

LEMMA 3.8.

(i) *Let $y \in C$ be a solution of* (3.3a) *with $f \in C$. Then*

$$Qy(0) = 0. \tag{3.26}$$

(ii) *Let y be a solution of* (3.4a) *with $f \in C$. Then $y \in C$ and*

$$Sy(0) = 0. \tag{3.27}$$

In both cases

$$My(0) = MRy(0) = 0.$$

[7]Note that $s^{I-M}(I-M)^{-1} = (I-M)^{-1}s^{I-M}$.

[8]Again, if $\sigma = 0$, then $\lambda = 0$.

The result of Lemma 3.8 means that in (3.3) the smoothness requirement (3.3c) is equivalent to $\mathrm{rank}(Q) = \mathrm{rank}(M) = n - \mathrm{rank}(R)$ homogeneous initial conditions, $Qy(0) = 0$ or $My(0) = 0$, the solution y must satisfy. In (3.4) the smoothness requirement (3.4c) is satisfied by any solution of (3.4a). Clearly, both statements are correct for the special spectral properties of M formulated in A.3.1 and A.3.2. We stress that these assumptions establish most general singular *initial value* problems of the form (3.3) and (3.4), where *all* conditions necessary and sufficient for a unique solution $y \in C$ have to be prescribed at one point, either at $t = 0$ or at $t = 1$.

THEOREM 3.9. *Let the $r \times r$ matrix $B_0\tilde{R}$ be nonsingular. Then for every $f \in C^p[0,1]$, $p \geq 0$, and any vector $\beta \in \mathbb{C}^r$, there is a unique solution $y \in C^{p+1}[0,1]$ of* (3.3)*. This solution has the form*

$$y(t) = \tilde{R}(B_0\tilde{R})^{-1}\beta + t\int_0^1 s^{-M} f(st)\,ds. \tag{3.28}$$

Furthermore,

$$|y(t)| \leq \text{const.}\, t\|f\| + |\tilde{R}(B_0\tilde{R})^{-1}\beta|, \tag{3.29}$$

$$|y'(t)| \leq \text{const.}\|f\|. \tag{3.30}$$

THEOREM 3.10. *Let B_1 be nonsingular. Then for every $f \in C^p[0,1]$, $p \geq 0$, and any vector $\beta \in \mathbb{C}^n$, there is a unique solution $y \in C[0,1] \cap C^{p+1}(0,1]$ of* (3.4)*. This solution is given by*

$$y(t) = t^M B_1^{-1}\beta + t^M \int_1^t \tau^{-M} f(\tau)\,d\tau, \tag{3.31}$$

and the following estimates hold:

$$|y(t)| \leq \begin{cases} \text{const.}\,(t^{\sigma_+}(1 + |\ln(t)|^{n_{\max}-1}) + 1)(|B_1^{-1}\beta| + \|f\|), & \sigma_+ < 1, \\ \text{const.}\,(t(1 + |\ln(t)|^{n_{\max}}) + 1)(|B_1^{-1}\beta| + \|f\|), & \sigma_+ = 1, \\ \text{const.}\,(t + 1)(|B_1^{-1}\beta| + \|f\|), & \sigma_+ > 1, \end{cases}$$

$$|y'(t)| \leq \begin{cases} \text{const.}\,(t^{\sigma_+ - 1}(1 + |\ln(t)|^{n_{\max}-1})(|B_1^{-1}\beta| + \|f\|) + \|f\|), & \sigma_+ < 1, \\ \text{const.}\,(1 + |\ln(t)|^{n_{\max}})(|B_1^{-1}\beta| + \|f\|), & \sigma_+ = 1, \\ \text{const.}\,(|B_1^{-1}\beta| + \|f\|), & \sigma_+ > 1. \end{cases}$$

Moreover, if $p < \sigma_+ \leq p + 1$, then $y \in C^p[0,1] \cap C^{p+1}(0,1]$ and if $\sigma_+ > p + 1$, then $y \in C^{p+1}[0,1]$.

In the next section we deal with problems with variable coefficient matrices $M(t)$.

3.2. Linear problems with variable coefficient matrix $M(t)$. Here we study initial value problems of the form

$$y'(t) = \frac{M(t)}{t}y(t) + f(t), \quad t \in (0,1], \tag{3.32a}$$

$$B_0 y(0) = \beta, \tag{3.32b}$$

$$My(0) = 0, \tag{3.32c}$$

where $M := M(0)$, and

$$y'(t) = \frac{M(t)}{t} y(t) + f(t), \quad t \in (0,1], \tag{3.33a}$$

$$B_1 y(1) = \beta. \tag{3.33b}$$

We discuss two cases, $M \in C[0,1] \cap C^1(0,1]$ and $M \in C^1[0,1]$, where M is chosen to have the form

$$M(t) = M + t^\gamma \mathring{C}(t), \quad \gamma > 0, \quad \mathring{C} \in C[0,1], \tag{3.34}$$

and

$$M(t) = M + t\,\mathring{C}(t), \quad \mathring{C} \in C[0,1], \tag{3.35}$$

respectively.

3.2.1. Coefficient matrix $M(t) \in C^1(0,1]$. Consider (3.32) with $M(t)$ given by (3.34) and assume that $M(0)$ satisfies A.3.1. Then (3.32a) is equivalent to

$$y'(t) = \frac{M}{t} y(t) + t^{\gamma-1}\mathring{C}(t) y(t) + f(t), \quad t \in (0,1]. \tag{3.36}$$

THEOREM 3.11. *If $B_0\tilde{R}$ is nonsingular, then for every $f \in C$ and $\mathring{C} \in C$, there exists a unique, continuous solution of* (3.32). *This solution satisfies $y \in C^1(0,1]$.*

Proof. According to Theorem 3.9 any continuous solution of the initial value problem (3.36) satisfies for $t \in [0,\delta]$,

$$y(t) = (\mathcal{KC}y)(t) + \psi(t), \tag{3.37}$$

where

$$(\mathcal{KC}y)(t) = t^\gamma \int_0^1 s^{-M} s^{\gamma-1} \mathring{C}(st) y(st)\, ds$$

and

$$\psi(t) = \tilde{R}(B_0\tilde{R})^{-1}\beta + t\int_0^1 s^{-M} f(st)\, ds.$$

Hence y is a fixed point of the operator $(\mathcal{KC}_\psi y)(t) := (\mathcal{KC}y)(t) + \psi(t)$, where $\mathcal{KC} : C[0,\delta] \to C[0,\delta]$ is a bounded linear operator with

$$\|\mathcal{KC}y\|_\delta \le D\,\delta^\gamma \|y\|_\delta, \quad D = \text{const.},$$

see Lemma 3.2. Therefore, the operator $\mathcal{KC}_\psi$ is contracting for δ sufficiently small, $\delta < (\frac{1}{D})^{\frac{1}{\gamma}}$. Now the Banach Fixed Point Theorem implies that there exists a unique solution $y \in C[0,\delta]$ of (3.37). This solution satisfies the initial condition (3.32b) and can be continued uniquely to $t = 1$. Finally, we substitute y into (3.36) and see that the structure of its first derivative is

$$y'(t) = t^{\gamma-1}\xi(t) + f(t), \quad \xi \in C.$$

Hence, $y \in C^1(0,1]$, and this completes the proof. ☐

We now estimate $y(t)$ for $t \in [0,\delta]$, where δ is chosen such that $\|\mathcal{KC}\|_\delta = L < 1$. Since

$$(\mathcal{I} - \mathcal{KC})^{-1} = \mathcal{I} + \sum_{i=1}^{\infty} (\mathcal{KC})^i,$$

we have

$$y(t) = \Big((\mathcal{I} - \mathcal{KC})^{-1}\psi\Big)(t) = \psi(t) + \sum_{i=1}^{\infty} \Big((\mathcal{KC})^i\psi\Big)(t).$$

Using Lemma 3.2 we now derive a bound for ψ,

$$\|\psi\|_\delta \le |\tilde{R}(B_0\tilde{R})^{-1}\beta| + \text{const.}\,\|f\|_\delta,$$

and by the Banach Lemma we conclude

$$\|y\|_\delta \le \frac{1}{1-L}\|\psi\|_\delta \le \text{const.}\,(|\tilde{R}(B_0\tilde{R})^{-1}\beta| + \|f\|_\delta).$$

It is quite plausible that such a uniform bound for y exists and its form is not surprising, either. It seems to be more interesting, though, to describe the local behavior of y by deriving an estimate for $y(t)$ for t close to zero. This is done by applying Lemma 3.2 and using the above bounds for $\|\psi\|_\delta$ and $\|y\|_\delta$ in (3.37),

$$|y(t)| \le \text{const.}\, t^\gamma(\|f\|_\delta + |\tilde{R}(B_0\tilde{R})^{-1}\beta|) + |\tilde{R}(B_0\tilde{R})^{-1}\beta|. \tag{3.38}$$

We now turn to (3.33) and assume that $M(0)$ satisfies A.3.2. For a nonsingular matrix B_1, the classical theory yields the existence of a unique solution $z(t)$ of (3.33), $z \in C[\delta,1]$, $0 < \delta \le 1$. Define $z(\delta) =: \omega$. In the following theorem we answer the question whether such a solution can be extended to $t = 0$.

THEOREM 3.12. *If B_1 is nonsingular, then for any $f \in C$ and $\mathring{C} \in C$, there exists a unique, continuous solution y of* (3.33). *This solution satisfies $y \in C^1(0,1]$.*

Proof. Consider (3.33a) for $t \in (0,\delta]$. Then, according to Theorem 3.10, any solution of (3.33a) subject to $y(\delta) = \omega$ satisfies

$$y(t) = (\mathcal{KC}y)(t) + \psi(t), \tag{3.39}$$

where

$$(\mathcal{KC}y)(t) = \int_\delta^t \left(\frac{t}{\tau}\right)^M \tau^{\gamma-1}\mathring{C}(\tau)y(\tau)\,d\tau$$

and

$$\psi(t) = \left(\frac{t}{\delta}\right)^M \omega + \int_\delta^t \left(\frac{t}{\tau}\right)^M f(\tau)\,d\tau.$$

Again, y is a fixed point of the operator $(\mathcal{KC}_\psi y)(t) := (\mathcal{KC}y)(t) + \psi(t)$, where the integral operator $\mathcal{KC} : C[0,\delta] \to C[0,\delta]$ is linear and bounded. We now use Lemma 3.6 to estimate $\mathcal{KC}y$. For $\sigma = \gamma$ we obtain

$$\|\mathcal{KC}y\|_\delta \le D\,\delta^{\gamma/2}\|y\|_\delta, \quad D = \text{const.}$$

on noting that $|t^{\gamma/2}\left(1+\left|\ln\left(\frac{t}{\delta}\right)\right|^n\right)|$ is bounded uniformly in $t\in[0,\delta]$. For the cases $\sigma<\gamma$, $\sigma>\gamma$, and for the contribution associated with $\lambda=0$ we have

$$\|\mathcal{K}\mathcal{C}y\|_\delta \le D\,\delta^\gamma \|y\|_\delta, \quad D = \text{const.}$$

and therefore $\mathcal{K}\mathcal{C}_\psi$ is contracting for a sufficiently small δ. □

Estimates for the solution and its first derivative, analogous to those in Theorem 3.10, can be derived also for the above case. We omit them to avoid unnecessary repetitions.

3.2.2. Coefficient matrix $M(t)\in C^1[0,1]$. In this section we study initial value problems (3.32) and (3.33), where $M(t)$ is given by (3.35). The corresponding system of differential equations reads

$$y'(t) = \frac{M}{t}y(t) + \mathring{C}(t)y(t) + f(t), \quad t\in(0,1] \tag{3.40}$$

which is equivalent to $\gamma=1$ in (3.36). The existence of continuous and unique solutions of (3.32) and (3.33) is obvious from the considerations in the previous section (with $\gamma=1$). Therefore, in the following two theorems we discuss in more detail only the smoothness properties of y.

THEOREM 3.13. *If $B_0\tilde{R}$ is nonsingular, then for every f, $\mathring{C}\in C^p$, $p\ge 0$, there exists a unique solution y of* (3.32). *This solution satisfies $y\in C^{p+1}[0,1]$.*

Proof. In (3.37) we set $\gamma=1$ and conclude that any continuous solution of (3.32) has the form

$$y(t) = t\eta(t) + \tilde{R}(B_0\tilde{R})^{-1}\beta, \quad \eta(t)\in C,$$

where

$$\eta(t) = \int_0^1 s^{-M}\mathring{C}(st)y(st)\,ds + \int_0^1 s^{-M}f(st)\,ds.$$

We now substitute y into (3.40). Hence,

$$y'(t) = M\eta(t) + \mathring{C}(t)y(t) + f(t)\in C$$

and $y\in C^1$. The result follows by successively applying the above argument in Theorem 3.9. □

THEOREM 3.14. *If B_1 is nonsingular, then for any f, $\mathring{C}\in C^p$, $p\ge 0$, there exists a unique solution y of* (3.33). *Moreover, if $p<\sigma_+\le p+1$, then $y\in C^p[0,1]\cap C^{p+1}(0,1]$ and if $\sigma_+>p+1$, then $y\in C^{p+1}[0,1]$.*

Proof. The smoothness results can be shown using techniques developed in Lemma 3.7. □

3.3. Nonlinear problems. In this section we discuss nonlinear problems of the form

$$y'(t) = \frac{M(t)}{t}y(t) + f(t,y(t)), \quad t\in(0,1], \tag{3.41a}$$

$$B_0y(0) = \beta, \tag{3.41b}$$

$$My(0) = 0, \tag{3.41c}$$

and

$$y'(t) = \frac{M(t)}{t} y(t) + f(t, y(t)), \quad t \in (0, 1], \tag{3.42a}$$

$$B_1 y(1) = \beta, \tag{3.42b}$$

where $f(t, y)$ is assumed to be continuous and Lipschitz-continuous with respect to y on a suitably defined domain.

We restrict our attention to the case where $M \in C^1[0, 1]$ is of the form (3.35) and study the system

$$y'(t) = \frac{M}{t} y(t) + \overset{\circ}{C}(t) y(t) + f(t, y(t)) =: \frac{M}{t} y(t) + F(t, y(t)). \tag{3.43}$$

Here, we assume that all quantities except for the eigenvalues of M are real.

THEOREM 3.15. *Let $f \in C^p([0, 1] \times \mathbb{R}^n)$, $\overset{\circ}{C} \in C^p[0, 1]$, $p \geq 0$, and let the matrix $B_0 \tilde{R}$ be nonsingular. Assume $f(t, y)$ to be Lipschitz-continuous with respect to y on $[0, 1] \times \mathbb{R}^n$. Then there exists a unique solution y of* (3.41) *and $y \in C^{p+1}[0, 1]$.*

Proof. We first prove the result on the subinterval $[0, \delta]$. Then standard arguments yield the extension to the whole interval, see §3.2.

Clearly, solving (3.41) on $[0, \delta]$ is equivalent to finding a fixed point of the nonlinear operator $(\mathcal{KF}_\gamma y)(t) := (\mathcal{KF} y)(t) + \gamma$, or equivalently, solving the nonlinear integral equation

$$y(t) = (\mathcal{KF} y)(t) + \gamma, \quad t \in [0, \delta], \tag{3.44}$$

where

$$(\mathcal{KF} y)(t) = t \int_0^1 s^{-M} F(st, y(st))\, ds = t \int_0^1 s^{-M} \left(\overset{\circ}{C}(st) y(st) + f(st, y(st)) \right) ds$$

and

$$\gamma = \tilde{R}(B_0 \tilde{R})^{-1} \beta.$$

From Lemma 3.2 we conclude that for a sufficiently small δ the operator $\mathcal{KF}_\gamma$ is contracting (with constant $L < 1$) on $C[0, \delta]$. Consequently, there exists a unique solution y of (3.44) and $y \in C[0, \delta]$. The smoothness of higher derivatives of y, $y \in C^{p+1}[0, 1]$, can be shown in a manner indicated in Theorem 3.13. □

Finally, we estimate y and y'. From

$$\|\mathcal{KF}_\gamma y\|_\delta - \|\mathcal{KF}_\gamma 0\|_\delta \leq \|\mathcal{KF}_\gamma y - \mathcal{KF}_\gamma 0\|_\delta \leq L \|y\|_\delta$$

we obtain

$$\|y\|_\delta \leq \frac{1}{1 - L} \left(|\gamma| + D \max_{\tau \in [0, \delta]} |f(\tau, 0)| \right) =: r.$$

We can now estimate f on the bounded domain

$$U := [0, \delta] \times \{ y \in \mathbb{R}^n : |y| \leq r \}$$

and define

$$F_\delta := \max_{(t,y)\in U} |f(t,y)|.$$

Using this bound and Lemma 3.2 we obtain for $t \in [0,\delta]$

$$|y(t)| \le \text{const.}\, t(F_\delta + |\tilde{R}(B_0\tilde{R})^{-1}\beta|) + |\tilde{R}(B_0\tilde{R})^{-1}\beta|, \tag{3.45}$$

$$|y'(t)| \le \text{const.}\, (F_\delta + |\tilde{R}(B_0\tilde{R})^{-1}\beta|). \tag{3.46}$$

For weaker smoothness assumptions on f, $f \in C^p([0,1] \times \{y \in \mathbb{R}^n : |y| \le r^*\})$ with $r^* \ge r$, the arguments of Theorem 3.15 can be used to show the existence, uniqueness and smoothness of y on $[0,\delta]$. In this case, though, the classical theory does not answer the question if this solution can be extended to the whole interval $[0,1]$.

THEOREM 3.16. *Let $f \in C^p([0,1] \times \mathbb{R}^n)$, $\mathring{C} \in C^p[0,1]$, $p \ge 0$, and let the matrix B_1 be nonsingular. Moreover, assume $f(t,y)$ to be Lipschitz-continuous with respect to y on $[0,1] \times \mathbb{R}^n$. Then there exists a unique solution y of (3.42). This solution satisfies $y \in C[0,1] \cap C^{p+1}(0,1]$. If $p < \sigma_+ \le p+1$, then $y \in C^p[0,1] \cap C^{p+1}(0,1]$, and if $\sigma_+ > p+1$, then $y \in C^{p+1}[0,1]$.*

Proof. The existence result follows by representing the solution in a way indicated in Theorem 3.12[9] and applying techniques from Theorem 3.15 with Lemma 3.6 instead of Lemma 3.2. The smoothness results can be shown in a way presented in Theorem 3.10. □

We note that for the solution of (3.41), $y \in C^1[0,1]$, the value of $y'(0)$ can be calculated by using the local Taylor expansion at $t = 0$. From

$$y(t) = y(0) + t\int_0^1 y'(\tau t)\, d\tau$$

and (3.41a) we have

$$\begin{aligned}\lim_{t\to 0} y'(t) &= \lim_{t\to 0}\left(\frac{1}{t}M(t)y(0) + M(t)\int_0^1 y'(\tau t)\, d\tau + f(t,y(t))\right)\\ &= \mathring{C}(0)y(0) + M(0)y'(0) + f(0,y(0)),\end{aligned}$$

which implies[10]

$$y'(0) = (I - M(0))^{-1}\Big(M'(0)y(0) + f(0,y(0))\Big). \tag{3.47}$$

The above representation does not hold for the solution of (3.42), since in this case $y'(0)$ may not exist.

4. Analytic results for systems of second order.

We now discuss the following classes of second order problems:

$$y''(t) = \frac{A_1(t)}{t}y'(t) + \frac{A_0(t)}{t^2}y(t) + f(t,y(t)), \quad t \in (0,1], \tag{4.1a}$$

$$B_0 y(0) = \beta, \tag{4.1b}$$

$$y \in C[0,1], \tag{4.1c}$$

[9] Here, by classical theory, a solution on $[\delta,1]$ exists.

[10] Note that $M(0)$ has only nonpositive eigenvalues and hence, $I - M(0)$ is nonsingular.

where B_0 is an $m \times n$ matrix and β is a vector of dimension $m \le n$, and

$$y''(t) = \frac{A_1(t)}{t} y'(t) + \frac{A_0(t)}{t^2} y(t) + f(t, y(t)), \quad t \in (0,1], \tag{4.2a}$$

$$B_0 y(1) + B_1 y'(1) = \beta, \tag{4.2b}$$

$$y \in C[0,1], \tag{4.2c}$$

where B_0, B_1 are $2n \times n$ matrices and β is a vector of dimension $2n$. We use the linear transformation $z(t) = (z_1(t), z_2(t))^T := (y(t), ty'(t))^T$ to transform the second order system to the first order form and obtain

$$z'(t) = \frac{M(t)}{t} z(t) + t\overset{\circ}{f}(t, z(t)), \quad t \in (0,1], \tag{4.3}$$

where

$$M(t) = \begin{pmatrix} 0 & I \\ A_0(t) & I + A_1(t) \end{pmatrix}, \quad \overset{\circ}{f}(t, z(t)) = \begin{pmatrix} 0 \\ f\,(t, z_1(t)) \end{pmatrix}. \tag{4.4}$$

Clearly, techniques and results from §3 can now be utilized in the investigations of (4.1) and (4.2). Notation is used accordingly. Especially, we again associate assumptions A.3.1 and A.3.2. with the matrix $M := M(0)$ occurring in the transformed versions of problems (4.1) and (4.2), respectively.

Linear problems with constant coefficient matrices A_0 and A_1. As a first step in the analysis of second order systems we study the linear problem

$$y''(t) = \frac{A_1}{t} y'(t) + \frac{A_0}{t^2} y(t) + f(t), \quad t \in (0,1], \tag{4.5a}$$

$$B_0 y(0) = \beta, \tag{4.5b}$$

$$A_0 y(0) = 0, \;\; y'(0) = 0. \tag{4.5c}$$

THEOREM 4.1. *Let $m = r$ and assume that the $r \times r$ matrix $B_0 I_1 \tilde{R}$ is nonsingular. Then for every $f \in C^p[0,1]$, $p \ge 0$, there is a unique continuous*[11] *solution of (4.5). This solution has the form*

$$y(t) = I_1 \tilde{R}(B_0 I_1 \tilde{R})^{-1} \beta + t^2 I_1 \int_0^1 s s^{-M} \overset{\circ}{f}(st)\, ds, \tag{4.6}$$

$$y'(t) = t I_2 \int_0^1 s s^{-M} \overset{\circ}{f}(st)\, ds. \tag{4.7}$$

Moreover, $y \in C^{p+2}[0,1]$, and the following estimates hold:

$$|y(t)| \le \text{const.}\, t^2 \|f\| + |\tilde{R}(B_0 I_1 \tilde{R})^{-1} \beta|, \tag{4.8}$$

$$|y'(t)| \le \text{const.}\, t \|f\|, \tag{4.9}$$

$$|y''(t)| \le \text{const.}\, \|f\|. \tag{4.10}$$

[11] Conditions (4.5c) are necessary and sufficient for the solution of (4.5a) to be continuous.

Proof. Consider the first order problem

$$z'(t) = \frac{M}{t} z(t) + t\mathring{f}(t), \quad t \in (0,1], \tag{4.11a}$$

$$B_0 z_1(0) = \beta, \tag{4.11b}$$

where

$$M = \begin{pmatrix} 0 & I \\ A_0 & I + A_1 \end{pmatrix}, \quad \mathring{f}(t) = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}. \tag{4.12}$$

Then by Theorem 3.9 the only continuous solution of (4.11a) has the form

$$z(t) = \gamma + t^2 \int_0^1 s s^{-M} \mathring{f}(st)\, ds,$$

where $\gamma \in \ker M = \mathcal{R}(R)$. We now choose γ in such a way that (4.11b) is satisfied. Note that the special form of M forces a certain structure of $\gamma = (\gamma_1, \gamma_2)^T$. Obviously, $M\gamma = 0$ implies $\gamma_2 = 0$. Let $\gamma_1 = I_1 \tilde{R}\alpha$, see the proof of Lemma 3.4 for notation, and substitute $z_1(0) = I_1 \tilde{R}\alpha$ into (4.11b). Then $\gamma_1 = I_1 \tilde{R}(B_0 I_1 \tilde{R})^{-1}\beta$ and hence the representations (4.6) and (4.7) follow. Condition $A_0 y(0) = 0$ is a consequence of $z(0) \in \ker M \Leftrightarrow Mz(0) = 0$ and $y'(0) = 0$ follows from (4.7).

Finally, from Theorem 3.9 we conclude $y \in C^{p+1}[0,1]$ and the substitution of (4.6) and (4.7) into (4.5a) yields $y \in C^{p+2}[0,1]$. All estimates can be derived in a straightforward manner. □

Note that $\gamma \in \ker M$ iff $\gamma_1 \in \ker A_0$ and $r = \text{rank}(R) = n - \text{rank}(A_0) = 2n - \text{rank}(M) \leq n$. This means in particular, that M nonsingular implies A_0 nonsingular. Consequently, if all real parts of the eigenvalues of M are negative, then the homogeneous initial condition $z(0) = 0$ for the first order system corresponds to $y(0) = 0$, $y'(0) = 0$ for the second order one, otherwise $Mz(0) = 0$ corresponds to $A_0 y(0) = 0$, $y'(0) = 0$. This set of equations provides $\text{rank}(A_0) + n = 2n - r$ linearly independent conditions to be satisfied for $y \in C$. The remaining r conditions necessary for the uniqueness of y are given by $B_0 y(0) = \beta$.

We now deal with the problem

$$y''(t) = \frac{A_1}{t} y'(t) + \frac{A_0}{t^2} y(t) + f(t), \quad t \in (0,1], \tag{4.13a}$$

$$B_0 y(1) + B_1 y'(1) = \beta, \tag{4.13b}$$

and formulate the standard result in the following theorem.

THEOREM 4.2. *Let the $2n \times 2n$ matrix $B_0 B_1$ be nonsingular. Then for every $f \in C^p[0,1]$, $p \geq 0$, there is a unique solution y of (4.13) and $y \in C[0,1] \cap C^{p+2}(0,1]$. This solution has the form*

$$y(t) = I_1 t^M \left((B_0 B_1)^{-1}\beta + \int_1^t \tau \tau^{-M} \mathring{f}(\tau)\, d\tau \right), \tag{4.14}$$

$$y'(t) = I_2 t^{M-I} \left((B_0 B_1)^{-1}\beta + \int_1^t \tau \tau^{-M} \mathring{f}(\tau)\, d\tau \right), \tag{4.15}$$

and the following estimates hold:

$$|y(t)| \leq \begin{cases} \text{const.}\,(t^{\sigma_+}(1+|\ln(t)|^{n_{\max}-1})+1)(|(B_0B_1)^{-1}\beta|+\|f\|), & \sigma_+ < 2, \\ \text{const.}\,(t^2(1+|\ln(t)|^{n_{\max}})+1)(|(B_0B_1)^{-1}\beta|+\|f\|), & \sigma_+ = 2, \\ \text{const.}\,(t^2+1)(|(B_0B_1)^{-1}\beta|+\|f\|), & \sigma_+ > 2, \end{cases}$$

$$|y'(t)| \leq \begin{cases} \text{const.}\,(t^{\sigma_+-1}(1+|\ln(t)|^{n_{\max}-1})(|(B_0B_1)^{-1}\beta|+\|f\|)+\|f\|), & \sigma_+ < 2, \\ \text{const.}\,t(1+|\ln(t)|^{n_{\max}})(|(B_0B_1)^{-1}\beta|+\|f\|), & \sigma_+ = 2, \\ \text{const.}\,t(|(B_0B_1)^{-1}\beta|+\|f\|), & \sigma_+ > 2, \end{cases}$$

$$|y''(t)| \leq \begin{cases} \text{const.}\,t^{\sigma_+-2}(1+|\ln(t)|^{n_{\max}-1})(|(B_0B_1)^{-1}\beta|+\|f\|), & \sigma_+ < 2, \\ \text{const.}\,(1+|\ln(t)|^{n_{\max}})(|(B_0B_1)^{-1}\beta|+\|f\|), & \sigma_+ = 2, \\ \text{const.}\,(|(B_0B_1)^{-1}\beta|+\|f\|), & \sigma_+ > 2, \end{cases}$$

where σ_+ is the smallest positive real part of the eigenvalues of M and $n_{\max}$ is the dimension of the largest Jordan box in the Jordan canonical form of M. Moreover, if $p < \sigma_+ \leq p+1$, then $y \in C^p[0,1] \cap C^{p+2}(0,1]$ and if $p+1 < \sigma_+ \leq p+2$, then $y \in C^{p+1}[0,1] \cap C^{p+2}(0,1]$. Finally, $y \in C^{p+2}[0,1]$ if $\sigma_+ > p+2$.

Proof. Theorem 3.10 and Lemma 3.6 with $\gamma = 2$. The estimates for y'' follow by combining the bounds derived separately for the components of the solution associated with the eigenvalue $\lambda = 0$ and the eigenvalues with positive real parts. □

Linear problems with variable coefficient matrices $A_0(t)$ and $A_1(t)$. We first consider the problem

$$y''(t) = \frac{A_1(t)}{t}y'(t) + \frac{A_0(t)}{t^2}y(t) + f(t), \quad t \in (0,1], \tag{4.16a}$$

$$B_0 y(0) = \beta, \tag{4.16b}$$

$$A_0 y(0) = 0, \;\; y'(0) = 0, \tag{4.16c}$$

where $A_0 := A_0(0)$. Motivated by the smoothness statements from Theorems 3.11 and 3.13, we assume $A_0, A_1 \in C^1[0,1]$ and write both matrices in the form

$$A_i(t) = A_i(0) + tC_i(t), \quad C_i \in C[0,1], \;\; i = 1,2. \tag{4.17}$$

Transformation to a first order problem yields

$$z'(t) = \frac{M}{t}z(t) + \mathring{C}(t)z(t) + t\mathring{f}(t), \tag{4.18}$$

where

$$M := \begin{pmatrix} 0 & I \\ A_0(0) & I + A_1(0) \end{pmatrix}, \quad \mathring{C}(t) := \begin{pmatrix} 0 & 0 \\ C_0(t) & C_1(t) \end{pmatrix}, \quad \mathring{f}(t) := \begin{pmatrix} 0 \\ f(t) \end{pmatrix}.$$

In the next theorem we need a finer structure of A_0 than of A_1 and therefore we additionally assume

$$A_0(t) = A_0(0) + tC_0(t) = A_0(0) + tA_0'(0) + t^2 D_0(t), \quad D_0 \in C. \tag{4.19}$$

THEOREM 4.3. *Let the matrix $B_0 I_1 \tilde{R}$ be nonsingular. Then for every $f, C_1 \in C^p[0,1]$ and $C_0 \in C^{p+1}[0,1]$, $p \geq 0$, there exists a unique solution y of (4.16) iff $A_0'(0)y(0) = 0$.*

This solution satisfies $y \in C^{p+2}[0,1]$. Moreover, there exists a $\delta > 0$, such that for every $t \in [0,\delta]$ the following estimates hold:

$$|y(t)| \le \text{const.}\, t^2(|\tilde{R}(B_0 I_1 \tilde{R})^{-1}\beta| + \|f\|_\delta) + |\tilde{R}(B_0 I_1 \tilde{R})^{-1}\beta|, \tag{4.20}$$

$$|y'(t)| \le \text{const.}\, t(|\tilde{R}(B_0 I_1 \tilde{R})^{-1}\beta| + \|f\|_\delta), \tag{4.21}$$

$$|y''(t)| \le \text{const.}\, (|\tilde{R}(B_0 I_1 \tilde{R})^{-1}\beta| + \|f\|_\delta). \tag{4.22}$$

Proof. Consider the problem

$$z'(t) = \frac{M}{t} z(t) + \overset{\circ}{C}(t) z(t) + t\overset{\circ}{f}(t), \quad t \in [0,\delta], \tag{4.23a}$$

$$B_0 z_1(0) = \beta. \tag{4.23b}$$

Define $\gamma := (\gamma_1, 0)^T \in \mathcal{R}(R)$, $\gamma_1 := I_1 \tilde{R}(B_0 I_1 \tilde{R})^{-1}\beta$[12]. It follows from Theorem 3.13 that for sufficiently small δ there exists a unique solution $z \in C^{p+1}[0,\delta]$ of (4.23). This solution has the form $z(t) = \gamma + t\zeta(t)$ and satisfies $Qz(0) = 0$. Substituting z into (4.18) gives

$$z'(t) = \frac{1}{t} M z(t) + \overset{\circ}{C}(t)\gamma + t(\overset{\circ}{C}(t)\zeta(t) + \overset{\circ}{f}(t)). \tag{4.24}$$

We conclude $y = z_1 \in C^{p+2}[0,\delta]$ on noting that $\overset{\circ}{C}(t)\gamma = (0, C_0(t)\gamma_1)^T \in C^{p+1}[0,\delta]$ and $\overset{\circ}{C}(t)\zeta(t) + \overset{\circ}{f}(t) \in C^p[0,\delta]$, see Theorem 3.13 and Theorem 4.1.

It is clear that in general $y'(0) \ne 0$, since from

$$z'(0) = (I - M)^{-1}\overset{\circ}{C}(0)\gamma$$

one only has[13]

$$y'(0) = -\Big(A_0(0) + A_1(0)\Big)^{-1} C_0(0) y(0).$$

In order to show that $A_0'(0)y(0) = 0$ is sufficient for y to satisfy $y'(0) = 0$ we rewrite $\overset{\circ}{C}(t)\gamma$,

$$\overset{\circ}{C}(t)\gamma = \begin{pmatrix} 0 & 0 \\ A_0'(0) & 0 \end{pmatrix}\gamma + t\begin{pmatrix} 0 & 0 \\ D_0(t) & \frac{C_1(t)}{t} \end{pmatrix}\gamma = t\overset{\circ}{D}(t)\gamma, \quad \overset{\circ}{D}(t) = \begin{pmatrix} 0 & 0 \\ D_0(t) & 0 \end{pmatrix},$$

and substitute the latter expression into (4.24),

$$z'(t) = \frac{1}{t} M z(t) + t(\overset{\circ}{D}(t)\gamma + \overset{\circ}{C}(t)\zeta(t) + \overset{\circ}{f}(t)).$$

Consequently,

$$z(t) = \gamma + t^2\vartheta(t), \quad y(t) = \gamma_1 + t^2 I_1 \vartheta(t), \quad y'(t) = t I_2 \vartheta(t),$$

[12] Note that $A_0(0)\gamma_1 = 0$.

[13] $(A_0(0) + A_1(0))$ is nonsingular because $(I - M)$ is nonsingular.

where

$$\vartheta(t) := \int_0^1 s s^{-M} \left(\overset{\circ}{D}(st)\gamma + \overset{\circ}{C}(st)\zeta(st) + \overset{\circ}{f}(st) \right) ds, \quad \vartheta \in C[0, \delta].$$

Clearly, the above y can be uniquely continued to $t = 1$ and the result follows. ◻

We stress that any continuous solution z of (4.23) satisfies $M(0)z(0) = 0$ and the corresponding conditions expressed in terms of y solving (4.16a) and (4.16b) read either

$$A_0(0)y(0) = 0, \quad \Big(A_0(0) + A_1(0)\Big)y'(0) + C_0(0)y(0) = 0,$$

or

$$A_0(0)y(0) = 0, \quad y'(0) = 0,$$

if $y(0) \in \ker A_0'(0)$.

We now discuss the problem

$$y''(t) = \frac{A_1(t)}{t} y'(t) + \frac{A_0(t)}{t^2} y(t) + f(t), \quad t \in (0, 1], \tag{4.25a}$$

$$B_0 y(1) + B_1 y'(1) = \beta, \tag{4.25b}$$

where $A_i(t)$ is given by (4.17). Then we have the following result.

THEOREM 4.4. *Let the $2n \times 2n$ matrix $B_0 B_1$ be nonsingular. Then for every $f, C_1 \in C^p[0, 1]$ and $C_0 \in C^{p+1}[0, 1]$, $p \geq 0$, there is a unique continuous solution of* (4.25). *Moreover, if $\sigma_+ > p + 1$, then $y \in C^{p+1}[0, 1]$, if $\sigma_+ > p + 2$, then $y \in C^{p+2}[0, 1]$.*

Proof. The existence and uniqueness of the solution z of the associated first order problem follow by applying Theorem 3.14 to (4.18).

To show the smoothness results, we first split[14] z,

$$z(t) = (R + S)z(t) = R(Rz(t)) + t\left(\frac{Sz(t)}{t}\right) =: Rr(t) + ts(t),$$

where $s \in C[0, 1]$ since $\sigma_+ > 1$. This implies the following structure of (4.18):

$$z'(t) = \frac{1}{t} M z(t) + \overset{\circ}{C}(t)Rr(t) + t(\overset{\circ}{C}(t)s(t) + \overset{\circ}{f}(t)).$$

Let $\sigma_+ > p + 1$, then

$$\overset{\circ}{C}(t)Rr(t) = \begin{pmatrix} 0 & 0 \\ C_0(t)I_1 R & 0 \end{pmatrix} r(t) \in C^{p+1}[0, 1]$$

and $\overset{\circ}{C}(t)s(t) + \overset{\circ}{f}(t) \in C^p[0, 1]$ and hence, the smoothness results follow from Theorem 3.14 and Theorem 4.2. ◻

[14] Recall that S is the projection onto X_+.

Nonlinear problems. Consider the problems

$$y''(t) = \frac{A_1(t)}{t} y'(t) + \frac{A_0(t)}{t^2} y(t) + f(t, y(t)), \quad t \in (0,1], \tag{4.26a}$$

$$B_0 y(0) = \beta, \tag{4.26b}$$

$$A_0 y(0) = 0, \quad y'(0) = 0 \tag{4.26c}$$

and

$$y''(t) = \frac{A_1(t)}{t} y'(t) + \frac{A_0(t)}{t^2} y(t) + f(t, y(t)), \quad t \in (0,1], \tag{4.27a}$$

$$B_0 y(1) + B_1 y'(1) = \beta. \tag{4.27b}$$

We transform the second order system to the first order form and have

$$z'(t) = \frac{M}{t} z(t) + \mathring{C}(t) z(t) + t\mathring{f}(t, z(t)), \tag{4.28}$$

where

$$M := M(0) = \begin{pmatrix} 0 & I \\ A_0(0) & I + A_1(0) \end{pmatrix}, \quad \mathring{C}(t) = \begin{pmatrix} 0 & 0 \\ C_0(t) & C_1(t) \end{pmatrix},$$

$$\mathring{f}(t, z(t)) := \begin{pmatrix} 0 \\ f(t, I_1 z(t)) \end{pmatrix},$$

see (4.17).

The following results are consequences of Theorems 4.3 and 4.4 applied to (4.26) and (4.27), respectively, where (4.18) is replaced by (4.28) and the Lipschitz-condition for $\mathring{f}$ is used.

THEOREM 4.5. *Let $f \in C^p([0,1] \times \mathbb{R}^n), C_1 \in C^p[0,1], C_0 \in C^{p+1}[0,1]$*[15]*, $p \geq 0$, and let the matrix $B_0 I_1 \tilde{R}$ be nonsingular. Assume $f(t,y)$ to be Lipschitz-continuous with respect to y on $[0,1] \times \mathbb{R}^n$. Then there exists a unique solution y of* (4.26) *iff $A_0'(0)y(0) = 0$. This solution satisfies $y \in C^{p+2}[0,1]$.*

THEOREM 4.6. *Assume $f \in C^p([0,1] \times \mathbb{R}^n), C_1 \in C^p[0,1], C_0 \in C^{p+1}[0,1], p \geq 0$, and the matrix $B_0 B_1$ to be nonsingular. Assume $f(t,y)$ to be Lipschitz-continuous with respect to y on $[0,1] \times \mathbb{R}^n$. Then there exists a unique continuous solution y of* (4.27). *Moreover, $y \in C^{p+1}[0,1]$ if $\sigma_+ > p+1$, and $y \in C^{p+2}[0,1]$ if $\sigma_+ > p+2$.*

Finally, using Taylor's Theorem we can derive a representation for $y''(0)$, where $y \in C^2[0,1]$ is a solution of (4.26); see §3.3 for the analogous representation of $y'(0)$ for the solution of (3.42),

$$\left(I - A_1(0) - \frac{A_0(0)}{2} \right) y''(0) = \frac{A_0''(0)}{2} y(0) + f(0, y(0)). \tag{4.29}$$

This is a system of n linear equations for $y''(0)$ which is uniquely solvable iff the leading coefficient matrix is nonsingular.

When applying a numerical method to solve (3.41) or (4.26) one often needs to provide a discretization at $t = 0$ for the system (3.41a) or (4.26a), respectively. In most of

[15] Again, assume $A_0(t) = A_0(0) + tC_0(t) = A_0(0) + tA_0'(0) + t^2 D_0(t)$ with $D_0 \in C$.

the methods this involves an evaluation of the corresponding right hand side at $t = 0$ which is not possible when the singularity is present. In such a case the local behavior of $y'(0)$ or $y''(0)$ described in (3.47) and (4.29), respectively, can be used to remedy the difficulty (see [9], [34]).

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